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Global and local existence of solution for fractional heat equation in $\mathbb{R}^N$ by Balakrishnan definition

Jorge Ferreira, Erhan Pişkin, Mohammad Shahrouzi*, Sebastião Cordeiro, Daniel V. Rocha

Abstract. Our aim here is to collect and to compare two definitions of the fractional powers of non-negative operators that can be found in the literature; we will present the proof of an equivalence and compare properties of that notions in different approaches. Then we will apply next this equivalence in the study of global and local existence of solution for the semilinear Cauchy problem in $\mathbb{R}^N$ with fractional Laplacian

$$\begin{cases} u_t = -(-\Delta)^\alpha u + f(x,u), \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N. \end{cases}$$

1. Introduction

When dealing with 'parabolic type' equations having fractional powers of sectorial operators in the main part (e.g. the $(-\Delta)^\alpha, \alpha \in (0,1)$), we face the situation that we need to work with different definitions of the fractional powers inside such considerations. In particular, proving local in time solvability, following Dan Henry's approach [3], we are using the Balakrishnan/Komatsu definition of the fractional power on non-negative operator. Studying next properties of such local solution, we need to use a Maximum Principle for fractional equation, that is based on another definition (3) of fractional powers through the singular integrals. Are the two definitions equivalent? And under which conditions. Are the two objects $(-\Delta)^\alpha$ obtained within the two definitions identical? Can we 'mixed' the two approaches within the studies? Due to increasing number of papers dealing with equations with fractional powers, like

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*Corresponding author.

$$\begin{cases} u_t = -(-\Delta)^{\alpha} u + f(x,u), \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{1}$$

such questions should be answered. Fortunately they have a positive answer and is given by Theorem 4.

The reader may easily find a increasing number of work making use of fractional powers of $-\Delta$ (like [14], and the other works of the same authors), however a rigorous and explicit proof of the equivalence described above has not been showed yet.

The first such type definition seems to be introduced for the square root of the m-accretive operator in Hilbert space (e.g. [5, Theorem 3.35]). A particular attention was also devoted to fractional powers of $-\Delta$ operator in $\mathbb{R}^N$, where the Fourier transform was used to introduce:

$$(-\Delta)^{-\frac{\alpha}{2}} f = \mathcal{F}^{-1}|x|^{-\alpha}\mathcal{F}f = \begin{cases} I^{\alpha} f, & \operatorname{Re}\alpha > 0, \\ D^{-\alpha} f, & \operatorname{Re}\alpha < 0. \end{cases}$$

Here f is a suitably regular function on $\mathbb{R}^N$, I denotes the *Riesz potential*:

$$I^{\alpha}\phi = \frac{1}{\gamma_N(\alpha)} \int_{\mathbb{R}^N} \frac{\phi(y)\,\mathrm{d}\,y}{|x-y|^{N-\alpha}}, \quad \alpha \neq N, N+2, N+4, \dots \tag{2}$$

and D^{α} is given by the hypersingular integral:

$$D^{\alpha} f = \frac{1}{d_{N,l}(\alpha)} \int_{|y|>\epsilon} \frac{(\Delta)_y^l f(x)}{|y|^{N+\alpha}}\,\mathrm{d}\,y, \tag{3}$$

see [9] for details.

A more general class of operators admitting fractional powers are the *sectorial positive operators*, e.g. [3, pg. 18]. Recall that a linear operator A in a Banach space X is called *sectorial* if it is closed and densely defined and its resolvent set contains the sector

$$S_{a,\phi} = \{\lambda; \phi \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\}. \tag{4}$$

Moreover, with certain $M \geq 1$, for all $\lambda \in S_{a,\phi}$ an estimate holds:

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - a|}. \tag{5}$$

A sectorial operator is called *positive* provided that $\operatorname{Re}\sigma(A) > 0$. In particular, a self-adjoint and positive definite operator in a Hilbert space satisfies the above requirement. A definition for fractional operator for this class of operators may be referred in [3].

In this work we will consider a larger class of operator, named *non-negative* operators, particularly the operator $-\Delta_p$, i.e., the distributional Laplacian in $L^p(\mathbb{R}^N)$, for which the definition by Balakrishnan/ Komatsu is satisfactory. Note that for this operator $0 \in \sigma(-\Delta_p)$.

During the decade of the 60s many authors in a large series of papers, presented concepts of power different to those of Balakrishnan, but equivalent to them, although using their own techniques and methodology. However, in that papers the powers were not located in a functional calculus for non-negative operators. Given a non-negative operator A, by a functional calculus associated to A we understand an application, in some sense homomorphic and continuous, which associates an operator $f(A)$ to every function belonging to an algebra of holomorphic functions. See [8] for a complete explanation.

One of the natural origin for equations including fractional-order derivative is the stochastic analysis; if the driven process is a jumps process (Levy process), then the corresponding Fokker-Planck equation will contain a fractional Laplacian. In this last years, in the study of fluid mechanics, finances, molecular biology and many other fields, it was discovered that the indraught of random factors can bring many new phenomena and features which are more realistic than the deterministic approach alone. Hence, it is natural to include stochastic terms, in particular fractional-order operators, when we establish the mathematical models. Moreover, the problems containing such fractional-order derivative terms becomes more challenging and many classical PDEs methods are hardly applicable directly to them, so that new ideas and theories are required.

To the best of our knowledge,this is the first work that dealing of (1) subject for a for fractional heat in $\mathbb{R}^N$ by Blakrishnan definition.

This paper is organized as follows: In Section 2, we present the definition and properties of fractional power of operator introduced by Balakrishnan/Komatsu [6, 7, 8]. In Section 3 is our main result, where we establish an equivalence between Balakrishnan definition and the other given by Fourier transform for positive powers of $-\Delta$ (Theorem 4). In Section 4, we use such different representations applying to the parabolic semilinear equation with fractional Laplacian (1).

2. Fractional powers of $-\Delta$

We most often use the definition of the fractional powers of operators by A.V. Balakrishnan in the form given in [6] (see also [10], p. 260). Consider X a Banach space and $A : D(A) \subset X \to X$.

Definition 1. We say that A is a non-negative operator if its resolvent set contains $]-\infty, 0[$ and the resolvent satisfies

$$\exists M > 0 \; \forall \lambda > 0 \quad \|A(\lambda + A)^{-1}\| \leq M.$$

Let $\alpha \in \mathbb{C}^+$ and an integer n such that $n-1 \leq \operatorname{Re}\alpha < n$. If A is densely defined, following Balakrishnan we define the power α of the operator A by

$$A^\alpha \phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty \lambda^{\alpha-1}[A(\lambda+A)^{-1}]^n \phi \, \mathrm{d}\,\lambda. \tag{6}$$

Since A is non-negative, the above integral converges absolutely.

Definition 2. We say that a closed linear operator $A : D(A) \subset X \to X$ is sectorial if A is non-negative and $\sigma(A)$ is contained in the closure of the sector

$$S_\omega = \{z \in \mathbb{C}^* : |\arg z| < \omega\}.$$

In this work we are interested in the fractional powers of the distributional Laplacian, $-\Delta_p$, $1 < p < \infty$, i.e,

$$\Delta_p f = \Delta f, \quad D(\Delta_p) = \{f \in L^p(\mathbb{R}^N) : \Delta_p f \in L^p(\mathbb{R}^N))\}.$$

As stated in [8], we have the following.

Theorem 1. *Let $1 < p < \infty$ and $\alpha > 0$. Then:*

(i) *$-\Delta_p$ is the infinitesimal generator of the heat semigroup, which is an analytic and contractive semigroup. Consequently $-\Delta_p$ is non-negative;*

(ii) *$(-\Delta_p)^\alpha$ is sectorial and $\sigma((-\Delta_p)^\alpha) = [0, \infty)$.*

For the next result, for $1 \leq p \leq \infty$ and $\alpha \in \mathbb{C}^+$, denote

$$S^{\alpha,p}(\mathbb{R}^N) := (D[(-\Delta_p)^{\alpha/2}], \| \cdot \|_{\alpha,p})$$

with $\| \cdot \|_{\alpha,p}$ being the graph norm of the closed operator $(-\Delta_p)^{\alpha/2}$.

Theorem 2. *If $1 < p < 1$, then $\mathcal{S}(\mathbb{R}^N)$ is dense in $S^{\alpha,p}(\mathbb{R}^N)$. Furthermore, if $0 < \alpha < m$, then $S^{\alpha,p}(\mathbb{R}^N) = [W^{m,p}(\mathbb{R}^N), L^p(\mathbb{R}^N)]_{\alpha/m}$ and their topologies coincide. Here $[W^{m,p}(\mathbb{R}^N), L^p(\mathbb{R}^N)]_{\alpha/m}$ stands for the space obtained by means of the complex interpolation method.*

Following the notation from [13, pages 219,222], denote

$$L^{s,p}(\mathbb{R}^N) := [W^{m,p}(\mathbb{R}^N), L^p(\mathbb{R}^N)]_{(m-s)/m},$$

with $s \leq m < s+1$. The next result is stated in [13].

Theorem 3. (i) *If $s \geq 0$ and $1 \leq p < \infty$, then $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{s,p}(\mathbb{R}^N)$.*

(ii) *If $t \leq s$ and if either $1 < p \leq q \leq Np/[N-(s-t)p] < \infty$ or $p = 1$ and $1 \leq q < N/(N-s+t)$, then $L^{s,t}(\mathbb{R}^N) \hookrightarrow L^{t,q}(\mathbb{R}^N)$.*

(iii) *$1 < p < \infty$ and $\epsilon > 0$, then for every s we have*

$$L^{s+\epsilon,p}(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^{s-\epsilon,p}(\mathbb{R}^N).$$

(iv) *If $s \geq 0$ and $1 \leq p \leq \infty$, then $L^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$.*

Remark 1. We also have this very useful immersions for fractional Sobolev spaces, known as net smoothness from [11]: if $1 < q \le p < \infty$ is such that $s - \frac{N}{p} \ge t - \frac{N}{q}$, then

$$W^{s,p}(\mathbb{R}^N) \subset W^{t,q}(\mathbb{R}^N), \quad s - \frac{N}{p} \ge t - \frac{N}{q}. \tag{7}$$

3. COMPARISON OF THE DEFINITIONS OF FRACTIONAL POWERS OF THE $(-\Delta)$ OPERATOR

As we mentioned before, another definition of fractional powers through the singular integrals is used when we formulate the Maximum Principle for fractional equation whereas for studying local solution for fractional parabolic equation via Dan Henry approach, we need the definition given by Balakrishnan/Komatsu. In this section we will compare these two definitions of the $(-\Delta)$ operator as introduced in the previous section, that is, we will show an equivalence between Balakrishnan definition of the positive power of $-\Delta$ and the one given by (3). We will give the rigorous proofs (known from the literature) of such equivalence. We also call the restrictions under which such an equivalence is possible.

The main result of this section is the equivalence that we present in the Theorem 4. It is also a generalization of the formulation found in [4].

Let $\mathcal{T}$ the family of complex functions defined on $\mathbb{R}^N$ such that any partial derivative belongs to $L^1 \cap L^\infty$, endowed with the natural topology defined by seminorms

$$|\phi|_m = \max\{\|D^\beta\phi\|_1, \|D^\beta\phi\|_\infty : |\beta| \le m\},$$

and consider the Laplacian $\Delta : \mathcal{T} \to \mathcal{T}$.

Lemma 1. *If $0 < \operatorname{Re}\alpha < N/2$, with $N \ge 2$, $\phi \in \mathcal{T}$ and $n > \operatorname{Re}\alpha \ge n-1$, then*

$$((-\Delta_{\mathcal{T}})^\alpha)(x) = \frac{\Gamma\left(\frac{N}{2} - (N-\alpha)\right)}{2^{2(n-\alpha)}\pi^{N/2}\Gamma(n-\alpha)}(|\cdot|^{2(n-\alpha)-N} * (-\Delta_{\mathcal{T}})^n\phi)(x), \tag{8}$$

for all $x \in \mathbb{R}^N$.

Proof. It is known that $\Delta_{\mathcal{T}}$ is infinitesimal generator of the C_0-semigroup $(P_t)_{t\ge 0}$, where $P_t\varphi = K_t * \varphi$, $\varphi \in \mathcal{T}$ and $K_t(x) = (4\pi t)^{-N/2}e^{-|x|^2/4t}$ (see [8, Th 2.5.1]). Consequently $\Delta_{\mathcal{T}}$ is a non-negative operator. The Balakrishnan operator for $-\Delta_{\mathcal{T}}$ leads us to

$$J^\alpha_{-\Delta_{\mathcal{T}}}\phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^\infty \lambda^{\alpha-1}[(\lambda - \Delta_{\mathcal{T}})^{-1}]^n(-\Delta_{\mathcal{T}})^n\phi \,\mathrm{d}\,\lambda.$$

By the Laplace transform applied to $[(\lambda - \Delta_{\mathcal{T}})^{-1}]^n\psi$ we get

$$[(\lambda - \Delta_{\mathcal{T}})^{-1}]^n\psi = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}(K_t * \psi)\,\mathrm{d}\,t \quad (\psi \in \mathcal{T}). \tag{9}$$

In particular if $\psi = (-\Delta_{\mathcal{T}})^n\phi$, then

$$(10)\quad (-\Delta_{\mathcal{T}})^\alpha\phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^\infty \lambda^{\alpha-1}\frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}(K_t*\psi)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda.$$

Noting that $\mathcal{T}-$convergence implies in pointwise convergence, it follows from (10) that

$$(11)\quad ((-\Delta_{\mathcal{T}})^\alpha\phi)(x) = \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^\infty \lambda^{\alpha-1}\left(\int_0^\infty t^{n-1}e^{-\lambda t}(K_t*\psi)(x)\,\mathrm{d}\,t\right)\mathrm{d}\,\lambda,$$

with $x\in\mathbb{R}^N$.

Using the identity

$$(12)\qquad \int_0^\infty \lambda^{\alpha-1}e^{-\lambda t}\,\mathrm{d}\,\lambda = \Gamma(\alpha)t^{-\alpha}$$

and the fact that $-\alpha+n-\frac{N}{2}<0$ and also the estimates

$$(13)\qquad |(K_t*\psi)(x)| \le \begin{cases} \|K_t\|_\infty\|\psi\|_1 = (4\pi t)^{-n/2}\|\psi\|_1, \\ \|K_t\|_1\|\psi\|_\infty = \|\psi\|_\infty. \end{cases}$$

we have by Tonelli theorem

$$\begin{aligned} ((-\Delta_{\mathcal{T}})^\alpha\phi)(x) &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\Gamma(\alpha)\int_0^\infty t^{-\alpha+n-1}(k_t*\psi)(x)\,\mathrm{d}\,t \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^\infty t^{-\alpha+n-1}\int_{\mathbb{R}^N}K_t(y)\psi(x-y)\,\mathrm{d}\,y\,\mathrm{d}\,t. \end{aligned}$$

Again by Tonelli theorem (and with the help of (13)), it follows

$$(14)\quad ((-\Delta_{\mathcal{T}})^\alpha\phi)(x) = \frac{1}{\Gamma(n-\alpha)}\int_{\mathbb{R}^N}\left(\int_0^\infty K_t(y)t^{-\alpha+n-1}\,\mathrm{d}\,t\right)\psi(x-y)\,\mathrm{d}\,y.$$

Now (8) follows from the relation

$$(15)\qquad \int_0^\infty K_t(y)t^{-\alpha+n-1}\,\mathrm{d}\,t = \frac{\Gamma\left(\frac{N}{2}-(n-\alpha)\right)}{2^{2(n-\alpha)}\pi^{N/2}}|y|^{2(n-\alpha)-N}.$$

The proof is complete. □

The well known technique to show an equivalence related to Balakrishnan operator of Laplacian is the Fourier transform. By the previous lemma, it will be very useful if we find a suitable space where we are allowed to calculate the Fourier transform of $|\cdot|^{2(n-\alpha)-N}$. Following the idea of Fourier transform in temperate distributions, we will establish a definition for a more restricted test function space by duality.

Let $\mathcal{S}(\mathbb{R}^N)$ be the set of smooth rapidly decreasing complex-valued functions on $\mathbb{R}^N$. We denote by

$$(16)\qquad \hat{\varphi}(x) = \mathcal{F}(\varphi)(x) = \int_{\mathbb{R}^N}\varphi(y)e^{ix\cdot y}\,\mathrm{d}\,y$$

the Fourier transform of $\varphi \in S(\mathbb{R}^N)$ and by

$$\tilde{\varphi}(y) = \mathcal{F}(\varphi)(y) = \int_{\mathbb{R}^N} \varphi(x) e^{iy\cdot x}\,\mathrm{d}\,x. \tag{17}$$

See [9] for these notations. Since $(-\Delta)^n \phi \in S(\mathbb{R}^N)$ when $\phi \in S(\mathbb{R}^N)$, $n \in \mathbf{N}$, its Fourier transform is well defined and is given by

$$\mathcal{F}((\Delta)^n \phi)(x) = -|x|^{2n}\mathcal{F}(\phi). \tag{18}$$

Now we will introduce a space of test functions called Lizorking space (see [9]), defined in the following manner

$$\Psi = \{\psi \in S(\mathbb{R}^N) : (D^j\psi)(0) = 0, |j| = 0, 1, 2, \dots\}. \tag{19}$$

Consider another spaces Φ of test functions as being the Fourier transforms of the elements from Ψ, i.e,

$$\Phi = \mathcal{F}(\Psi) = \{\phi \in S(\mathbb{R}^N) : \phi \in \hat{\psi}, \quad \psi \in \Psi\}. \tag{20}$$

If $g \in \Psi'$, we define its Fourier transform by the relation

$$(\hat{g}, \varphi) = (g, \hat{\varphi}), \quad \varphi \in \Phi. \tag{21}$$

The following lemma is stated in [9, pg. 490].

Lemma 2. *The Fourier transform of the function $|\cdot|^{-\alpha}$, interpreted according to (21) is given by the relation*

$$\mathcal{F}(|\cdot|^{-\alpha})(x) = \frac{(2\pi)^N}{\gamma_N(\alpha)} \begin{cases} |x|^{\alpha-N}, & \alpha \neq N+2k, \alpha \neq -2k, \\ |x|^{\alpha-N} \ln \frac{1}{|x|}, & \alpha = N+2k, \\ (-\Delta)^{-\alpha/2}\delta, & \alpha = 2k, \end{cases} \tag{22}$$

where $\delta = \delta(x)$ is the Dirac delta-function, $k = 0, 1, 2, \dots,$ the constant $\gamma_N(\alpha)$ being equal to

$$\gamma_N(\alpha) = \begin{cases} 2^\alpha \pi^{N/2}\Gamma(\frac{N}{2})/\Gamma(\frac{N-\alpha}{2}), \quad \alpha \neq N+2k, \alpha \neq -2k, \\ 1, \quad \alpha = -2k, \\ (-1)^{(N-\alpha)/2}\pi^{N/2}2^{\alpha-1}\left(\frac{\alpha-N}{2}\right)!\Gamma\left(\frac{N}{2}\right), \quad \alpha = N+2k. \end{cases} \tag{23}$$

Let us now consider the topological dual space $\mathcal{T}'$, endowed with the topology of uniform convergence on bounded sets of $\mathcal{T}$. By [8, Prop. 2.5.3], if $1 \leq p \leq \infty$, then $L^p \subset \mathcal{T}'$ and the usual topology on L^p is stronger than that $\mathcal{T}'$ induces on L^P. Define $\Delta_{\mathcal{T}'} : \mathcal{T}' \to \mathcal{T}'$ by the duality

$$((\Delta_{\mathcal{T}'})u, \phi) = (u, (\Delta_{\mathcal{T}})\phi) \quad (\phi \in \mathcal{T}, u \in \mathcal{T}'). \tag{24}$$

It follows from [8, Cor. 5.2.4] that

$$((\Delta_{\mathcal{T}'})^\alpha u, \phi) = (u, (\Delta_{\mathcal{T}})^\alpha \phi) \quad (\phi \in \mathcal{T}, u \in \mathcal{T}', \alpha \in \mathbb{C}^+). \tag{25}$$

From (11) we can check that if $u \in C_0^\infty(\mathbb{R}^N)$, then

$$(-\Delta_{\mathcal{T}'})^\alpha u = (-\Delta_{\mathcal{T}})^\alpha u, \tag{26}$$

where u in the left side of this equality has to be understood as an element of $\mathcal{T}'$. The relation between $(-\Delta_{\mathcal{T}'})^\alpha$ and $(-\Delta_p)^\alpha$, $1 \leq p \leq \infty$ is given by [8, Th 12.1.6] , i.e,

$$[(-\Delta_{\mathcal{T}'})^\alpha]_p = (-\Delta_p)^\alpha, \tag{27}$$

where $[(-\Delta_{\mathcal{T}'})^\alpha]_p$ is the part of $(-\Delta_{\mathcal{T}'})^\alpha$ in $L^p(\mathbb{R}^N)$.

Theorem 4. *Let* $1 < p < \infty$*,* $\alpha > 0$*,* $N \geq 2$ *and* $n > \alpha \geq n-1$ *such that* $\alpha < N/2$*,* $\alpha \neq (n-k) - N/2$*,* $k = 0, 1, 2, 3, \dots$*, then*

$$(-\Delta_p)^\alpha \phi = \mathcal{F}^{-1}(|\cdot|^{2\alpha}\hat{\phi}) \quad (\phi \in C_0^\infty(R^N)) \tag{28}$$

and

$$\mathcal{F}^{-1}(|\cdot|^{2\alpha}\hat{\phi})(x) = \frac{1}{d_{N,l}(2\alpha)} \int_{\mathbb{R}^N} \frac{(\Delta)_y^l \phi}{|y|^{N+2\alpha}} \, \mathrm{d}\, y \quad (\phi \in C_0^\infty(R^N)), \tag{29}$$

where $l > 2\alpha$ *(see* [9, Sec. 25.4] *for notations).*

Proof. Given $\phi \in C_0^\infty$, by (26) and Lemma 1

$$(-\Delta_{\mathcal{T}'})^\alpha \phi = \frac{\Gamma(N/2 - (n-\alpha))}{2^{2(n-\alpha)}\pi^{N/2}\Gamma(n-\alpha)}(|\cdot|^{2(n-\alpha)-N} * (-\Delta_{\mathcal{T}})^n \phi). \tag{30}$$

Since $\mathcal{T}' \subset \Psi'$ we can apply Fourier transform in the both sides. By Lemma 2, replacing α by $N - 2(n-\alpha)$, it follows

$$\mathcal{F}(|\cdot|^{2(n-\alpha)-N}) = \frac{2^{2(n-\alpha)}\pi^{N/2}\Gamma(n-\alpha)}{\Gamma(N/2 - (n-\alpha))}|\cdot|^{2\alpha-2n}. \tag{31}$$

Thus

$$\mathcal{F}[(-\Delta_{\mathcal{T}'})^\alpha \phi] = |\cdot|^{2\alpha}\hat{\phi}, \tag{32}$$

i.e,

$$(-\Delta_{\mathcal{T}'})^\alpha \phi = \mathcal{F}^{-1}(|\cdot|^{2\alpha}\hat{\phi}). \tag{33}$$

By [8, Cor. 12.3.5], the part of $(-\Delta_{\mathcal{T}'})^\alpha$ in $L^p(\mathbb{R}^N)$ is $[L^p(\mathbb{R}^N), W^{N,p}]_{\alpha/N}$, where this last notation stands for the space obtained by means of the complex interpolation method. It follows from (27) that $(-\Delta_p)^\alpha \phi = (-\Delta_{\mathcal{T}'})^\alpha \phi$ since $C_0^\infty(\mathbb{R}^N) \subset [L^p(\mathbb{R}^N), W^{N,p}]_{\alpha/N}$. This proves (28).

Now recalling the relations stated by [9], we have

$$\mathcal{F}(T^{2\alpha}\phi) = d_{N,l}(2\alpha)|\cdot|^{2\alpha}\hat{\phi} \quad (\phi \in C_0^\infty(R^N)), \tag{34}$$

where

$$T^{2\alpha}\phi = \int_{\mathbb{R}^N} \frac{(\Delta_y^l)\phi}{|y|^{N+2\alpha}} \, \mathrm{d}\, y, \tag{35}$$

i.e,

$$\frac{1}{d_{N,l}(2\alpha)} \int_{\mathbb{R}^N} \frac{(\Delta_y^l)\phi}{|y|^{N+2\alpha}}\,\mathrm{d}\,y = \mathcal{F}^{-1}(|\cdot|^{2\alpha}\hat{\phi}) \quad (\phi \in C_0^\infty(\mathbb{R}^N)). \tag{36}$$

The proof is complete. □

Corollary 1. *If $n \le \alpha < n+1$, then the equality (29) holds for $N \ge 2(n+1)$. If $\alpha = n$, its is enough that $N \ge 2n+1$.*

4. Application to exemplary semilinear equation with fractional power operator

We will show now the application of the results presented in the two following sections. Namely, consider the 'parabolic' semilinear Cauchy's problem with fractional Laplacian in the main part:

$$\begin{cases} u_t = -(-\Delta)^\alpha u + f(x,u), \quad t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{37}$$

Dan Henry's technique will be used first to set that problem locally, that means, to obtain a *local in time* solution of it in an appropriate phase space X^β? Then, when we want to extend such a local solution globally in time, an a priori estimate (sufficiently well) is needed for doing that. In case of the Cauchy problem (37) the role of that a priori estimate is played by a version of the *Maximum Principle* valid for a sufficiently regular solutions to (37). When proving that Maximum principle, we need to use another (precise, which one) definition of fractional power of $(-\Delta)$ and the important pointwise inequality from [2]. Hence, it is important to note that the two definitions of the fractional Laplacian lead to the same object.

The reader will find the definitions and theorems throughout this section in [1].

Definition 3. For each $\alpha \ge 0$, define $X^\alpha := D(A_1^\alpha)$ with the norm graph, i.e, $\|x\|_\alpha = \|A_1^\alpha\|$, $x \in X^\alpha$, where $A_1 = A + a$ with a chosen so that $\operatorname{Re}\sigma(A_1) > 0$.

Different choices of a give equivalent norms on X^α, so we suppress the dependence on choice of a. Furthermore X^α is a dense subset of X^β for $\alpha \ge \beta \ge 0$ with continuous inclusion.

Local solvability. Let $\alpha \in [0,1)$ and u_0 be an element of X^α. If, for some real $\tau > 0$, a function $u \in C([0,\tau), X^\alpha)$ satisfies

- $u(0) = u_0$,
- $u \in C^1((0,\tau), X^\alpha)$,
- $u(t) \in D(A)$ for each $t \in (0,\tau)$,
- the equation (37) holds in X for all $t \in (0,\tau)$,

then u is called a local X^α solution of (37).

Theorem 5. *Let $F : X^\alpha \to X$ be Lipschitz continuous on bounded subsets of X^α for some $\alpha \in [0, 1)$. For each $u_0 \in X^\alpha$, there exists a unique X^α solution $u = u(t, u_0)$ of (37) defined on its maximal interval of existence $[0, \tau_{u_0})$ which means that $\tau_{u_0} = \infty$, or if $\tau_{u_0} < \infty$ then $\limsup_{t\to\tau_{u_0}^-} \|u(t, u_0)\|_{X^\alpha} = +\infty$.*

Application for local solvability. Let $X = L^p(\mathbb{R}^N)$, $p > N$, be the base space for the Cauchy problem

$$\begin{cases} u_t = -(-\Delta_p)^\alpha u + F(u), & 1 > \alpha > 1/2, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{38}$$

where $F : u \mapsto u|u|^\nu$, $\nu > 1$.

Lets find $\beta \in (0, 1)$ for which the problem (38) has local X^β solution. It means that such β has to satisfy

$$\forall r > 0\ \forall \varphi, \psi \in B_{X^\beta}(r)\ \exists L(r)\ \ \|F(\varphi) - F(\psi)\| \leq L(r)\|\varphi - \psi\|_{X^\beta}. \tag{39}$$

Since $\sigma[I + (-\Delta_p)^\alpha] = [1, +\infty)$, we have $\sigma(A_1) > 0$ if $A_1 = I + (-\Delta_p)^\alpha$. Thus

$$X^\beta = D(A_1^\beta) = D[(1 + (-\Delta_p)^\alpha)^\beta] = D[(-\Delta_p)^{\alpha\beta}].$$

If $2\alpha\beta < 1$, $X^\beta \subset W^{s,p}(\mathbb{R}^N)$, where $s = 2\alpha\beta$. Let's check (39). Let $\varphi, \psi \in X^\beta$. Using Holder and

$$|\varphi|\varphi|^\nu - \psi|\psi|^\nu| \leq (\nu + 1)(|\varphi|^\nu + |\psi|^\nu)|\varphi - \psi|$$

inequalities, we obtain

$$\|F(\varphi) - F(\psi)\|_{L^p(\mathbb{R}^N)} \leq c(\|\varphi\|^\nu_{L^{pq\nu}(\mathbb{R}^N)} + \|\psi\|^\nu_{L^{pq\nu}(\mathbb{R}^N)})\|\varphi - \psi\|_{L^{pq^*}(\mathbb{R}^N)},$$

where $1/q + 1/q^* = 1$, provided that

$$L^{pq\nu}(\mathbb{R}^N) \supset W^{s,p}(\mathbb{R}^N), \tag{40}$$

$$L^{pq^*\nu}(\mathbb{R}^N) \supset W^{s,p}(\mathbb{R}^N). \tag{41}$$

The condition $F(W^{s,p}(\mathbb{R}^N)) \subset L^p(\mathbb{R}^N)$ implies

$$L^{(\nu+1)p}(\mathbb{R}^N) \supset W^{s,p}(\mathbb{R}^N). \tag{42}$$

Choosing $q \geq 2$, we get $s \geq \frac{N}{p}\frac{\nu}{\nu+1}$. Thus if $\frac{1}{2\alpha} > \beta \geq \frac{N}{p}\frac{\nu}{\nu+1}\frac{1}{2\alpha}$ the problem (38) has local X^β solution.

Global solvability. Now we shall find conditions to extend the local solution in the previous application.

A function $u = u(t)$ is called a global X^α solution of (37) if it fulfills the requirements of local solvability with $\tau = \infty$.

Theorem 6. *Global solvability of (37) follows if it is possible to choose*

- *a Banach space Y, with $D(A) \subset Y$,*
- *a locally bounded function $c : [0, +\infty) \to [0, +\infty)$,*
- *a nondecreasing function $g : [0, +\infty) \to [0, +\infty)$,*

– *a certain number $\theta \in [0,1)$ such that for each $u_0 \in X^\alpha$, both conditions*

$$\|u(t,u_0)\|_Y \leq c(\|u_0\|_{X^\alpha}), \;\; t \in (0,\tau_{u_0}) \tag{43}$$

and

$$\|F(u(t,u_0))\|_X \leq g(\|u(t,u_0)\|_Y)(1+\|u(t,u_0)\|^\theta_{X^\alpha}), \;\; t \in (0,\tau_{u_0}) \tag{44}$$

hold.

Application for Global solvability. Here we will deal with both definitions of $(-\Delta)^\alpha$. The one given by Fourier transform shows up in the lemma below, used to prove the a priori estimate for the Maximum Principle.

Lemma 3. *Suppose that $\alpha \in (0,1)$, $q \geq 2$ and $\varphi, (-\Delta)^\alpha\varphi \in L^q(\mathbb{R}^N)$. Then, the following inequality holds*

$$\int_{\mathbb{R}^N} |\varphi|^{q-2}\varphi(-\Delta)^\alpha\varphi dx \geq \frac{2}{q}\int_{\mathbb{R}^N}((-\Delta)^\alpha|\varphi|^{\frac{q}{2}})^2 \,\mathrm{d}\,x,$$

where the operator $(-\Delta)^\alpha$ is referred to the Fourier transform definition.

Proof. See [12]. □

The Maximum Principle a priori estimate is stated by the following lemma.

Lemma 4. *If $\theta(\cdot,\theta_0)$ is a local solution of (38), the following estimates:*

$$\|\theta(t,\theta_0)\|_{L^q(\mathbb{R}^N)} \leq \|\theta_0\|_{L^q(\mathbb{R}^N)}, \quad q \in [p,\infty] \tag{45}$$

holds.

Proof. Multiplying (38) by $|\theta|^{q-1}\operatorname{sgn}(\theta)$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\theta_t|\theta|^{q-1}\operatorname{sgn}(\theta)\,\mathrm{d}\,x = &-\int_{\mathbb{R}^N}(-\Delta_p)^\alpha\theta|\theta|^{q-1}\operatorname{sgn}(\theta)\,\mathrm{d}\,x\\ &+\int_{\mathbb{R}^N}F(\theta)|\theta|^{q-1}\operatorname{sgn}(\theta)\,\mathrm{d}\,x.\end{aligned}$$

Since the first term of right side is negative due to Maximum Principle a priori estimate (Lemma 3), we get

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\mathbb{R}^N}|\theta|^q\,\mathrm{d}\,x \leq -\int_{\mathbb{R}^N}|\theta|^q\,\mathrm{d}\,x.$$

Solving the above differential inequality we get (45) for $q \in [p,\infty)$. Take the limit $q \to \infty$, (45) also holds for $q = \infty$. □

For our application, consider the Cauchy problem (38), but supposing now that $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz with the restrictions $F(s)sgn(s) \leq -|s|$ and $|F(s)| \leq 2|s| + |s|^p$, $s \in \mathbb{R}$. Choosing the phase space X^β for which $1 > \beta > \frac{N}{p}$ we have the inclusion $X^\beta \subset L^\infty(\mathbb{R}^N)$ and therefore F is Lipschitz continuous on bounded sets as map from X^β to X, what implies in the existence of local X^β solutions. To show global in time extendibility of the

X^β solution, we need first to get a priori estimate of it in an auxiliary Banach space. We choose $Y = L^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Such a priori estimate is played by Lemma 4.5. As $\|\cdot\|_Y := \max\{\|\cdot\|_{L^p(\mathbb{R}^N)}, \|\cdot\|_{L^\infty(\mathbb{R}^N)}\}$, it follows from the previous lemma that (43) is satisfied. By the restriction $|F(s)| \le 2|s| + |s|^p$, there exists a positive constant C_0 such that

$$\begin{aligned}\|F(u))\|_{L^p(\mathbb{R}^N)} &\le \|u\|^{p-1}_{L^\infty(\mathbb{R}^N)}\|u\|_{L^p(\mathbb{R}^N)} + 2\|u\|_{L^p(\mathbb{R}^N)}\\ &\le C_0(\|u\|^p_Y + \|u\|_Y).\end{aligned}$$

Therefore, (44) also holds.

5. CONCLUSION

In this work, we obtained the local and global solutions for a fractional heat equation in R^n by Balakrishnan definition in an unbounded domain.

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Jorge Ferreira
Department of Exact Sciences
Federal Fluminense University
27255-435 Volta Redonda, RJ
Brazil
E-mail address: ferreirajorge2012@gmail.com

Erhan Pişkin
Department of Mathematics
Dicle University
TR-21280 Diyarbakir
Turkey
E-mail address: episkin@dicle.edu.tr

Mohammad Shahrouzi
Department of Mathematics
Jahrom University
Jahrom, P.O.Box: 74137-66171
Iran
E-mail address: mshahrouzi@jahromu.ac.ir

Sebastião Cordeiro
Faculty of Exact Sciences and Technology
Federal University of Pará
68440-000, Abaetetuba, Pará
Brazil
E-mail address: sebastiao@ufpa.br

Daniel V. Rocha
Institute of Exact and Natural Sciences
Federal University of Pará
Augusto corrêa Street, Number 01
66075-110, Belém PA
Brazil
E-mail address: danielvrocha2011@gmail.com